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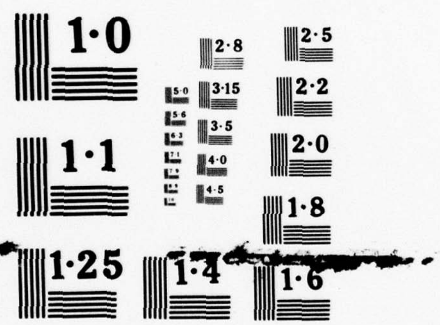

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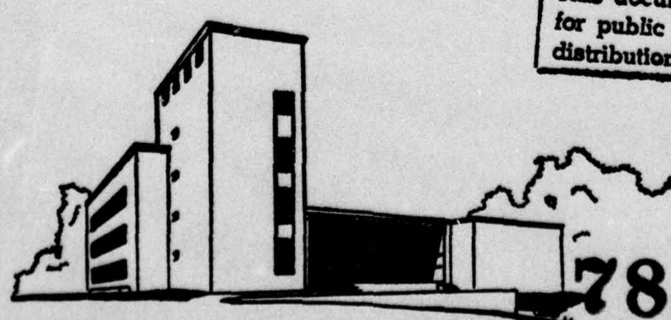

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A LIMITING LAGRANGEAN FOR INFINITELY-CONSTRAINED CONVEX OPTIMIZATION IN R^n.

by

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(12) 35p.

(11) Apr[redacted]78

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Abstract

R superscript n

It is shown, ~~We show~~, for convex optimization in R^n, how a minor modification of the usual Lagrangean function (unlike that of the "augmented Lagrangeans"), plus a limiting operation, allows one to close duality gaps even in the absence of a Kuhn-Tucker vector, (see the introductory discussion, and the discussion in Section 4 up to equation (50)). The cardinality of the convex constraining functions can be arbitrary (finite, countable, or uncountable).

In fact, the ~~our~~ main result ~~(Theorem 3 of Section 3)~~ reveals much finer detail concerning the ~~our~~ "Limiting Lagrangean." There are affine minorants

theta is ≤ or = to 1 theta

(for any value $0 < \theta \leq 1$ of the limiting parameter θ) of the given convex functions, plus an affine form nonpositive on K, for which a general

R superscript n

linear inequality holds on R^n. After substantial weakening, this inequality leads to the conclusions of the previous paragraph.

This work is motivated by, and is a direct outgrowth of, research joint with R.J. Duffin, which is cited as our reference [6].

Key Words:

1) Lagrangean
2) Nonlinear programming
3) Kuhn-Tucker theory
4) Convex function
5) Convexity

A Limiting Lagrangean for Infinitely-Constrained Convex Optimization in R^n

by R.G. Jeroslow[1]

In Honor of Dick Duffin

We consider convex programs (see Section 2 below)

$$\text{(CP)} \qquad \begin{array}{c} \inf f_0(x) \\ \text{subject to } f_h(x) \leq 0,\ h \in H, \\ x \in K \end{array}$$

with possibly infinitely many constraints, and show under a weak constraint qualification (CQ) (see below) that a small modification of the ordinary Lagrangean always closes the duality gap. To be more specific (see Corollary 3 below) we show there are scalars w_0, w_1 and a vector $w \in R^n$ such that, if $0 < \theta \leq 1$, there are nonnegative scalars $\{\lambda_h \mid h \in H\}$ with

$$\text{(DE)} \qquad (1 + \theta w_0)f_0(x) + \theta(wx + w_1) + \sum_{h \in H} \lambda_h f_h(x) \geq v(P)$$

for all $x \in K$, where $v(P)$ is the value (assumed finite) of (CP). The summation in (DE) is never problematic, since only finitely many λ_h are non-zero.

Our constraint qualification (CQ) does not imply the existence of a Kuhn-Tucker vector, and hence is weaker than the usual ones (see Section 2 below). Thus (DE) places many duality gaps in a simple perspective: the criterion function $f_0(x)$ should not be weighted by unity, but rather by a number arbitrarily near unity; and then an affine linear "compensation" $wx + w_1$ is needed, but it can be weighted by any positive amount, however small.

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Our methods of proof can be succinctly described, and were developed jointly in [6]. We reason as follows. Since closed convex sets, and the epigraphs of closed, convex functions, are describable by infinitely many linear inequalities — in the terminology of Charnes, Cooper, and Kortanek [3], they are describable as a "semi-infinite" constraint set — this convex optimization ought to, in principle, be reducible to results about semi-infinite systems.

Recently, R.J. Duffin and the author [6] found methods of reducing convex programs, under a constraint qualification, to semi-infinite programs, of applying the "appropriate" result on semi-infinite systems, and then re-interpreting the resulting conclusion (which is a conclusion about the semi-infinite program) as a conclusion about the convex program.

This paper is very much a replication of [6], except that a different result about semi-infinite programs is (first established here in *Section* 1 and then) applied, and then a different conclusion about the Lagrangean is obtained.

For related work, see Blair's generalization [2 , Theorem 3] of a result from an early draft of [6], which we quoted to him, as well as McLinden's further generalization of this result [8] to certain infinite-dimensional spaces. McLinden's work [8] uses the elegant theory of conjugates of convex functions, as developed by Rockafellar in [9]. The paper [4] is also relevant.

Section 1: A Strengthening of a Corollary of Blair's "Ascent Ray" Theorem

In this section, we strengthen [1 , Corollary 2] to a form which, as we show below, actually implies the main result [1 , Theorem] of [1].

Let cone(S) resp. clcone(S) denote the cone resp. the closure of the cone spanned by S (see [9]). The following result is well-known (see, e.g., [9]), and is a direct application of the Separating Hyperplane Theorem.

Lemma 1: For $I \neq \emptyset$ an arbitrary index set, indexing a set of vectors $\{a^i \mid i \in I\}$ in R^n, suppose that

$$a^i x \geq 0, \quad \text{all } i \in I$$

implies

$$cx \geq 0$$

for any $x \in R^n$.

Then $c \in \text{clcone}\,(\{a^i \mid i \in I\})$.

Lemma 2: For $I \neq \emptyset$ an arbitrary index set, suppose that

$$a^i x \geq 0, \quad \text{all } i \in I$$

implies

$$cx \geq 0.$$

Then there is a vector w with the following property:

For any θ, $0 < \theta \leq 1$, there is a set of nonnegative multipliers $\{\lambda_i \mid i \in I\}$, only finitely non-zero, such that

$$(1) \qquad c + \theta w = \sum_{i \in I} \lambda_i a^i.$$

In fact, if v is any point in the relative interior of the set

$$(2) \qquad C' = \text{cone}\,(\{a^i \mid i \in I\})$$

we may set

(3) $$w = v - c.$$

Proof. By Lemma 1, $c \in \text{cl}\, C'$, and since v is in the relative interior of C', then $0 < \theta \leq 1$ implies that $\theta v + (1-\theta)c$ is in the relative interior of C', by the accessibility lemma [9]; and hence can be expressed in the form of the right-hand-side of (1) with $\{\lambda_i \mid i \in I\}$ a finitely non-zero set of multipliers. However,

$$\theta v + (1-\theta)c \;=\; c + \theta(v-c) \;=\; c + \theta w$$

and so (1) holds.

Since any convex set C' has a relative interior, at least one such w given by (3) exists.

Q.E.D.

We now give our strengthening of [1 , Corollary 2], which is closely related to Kortanek's "perfect duality" results [7].

Theorem 1: Let $I \neq \emptyset$ be an arbitrary index set, and suppose that the system

(4) $$a^i x \geq b \;, \quad \text{all } i \in I$$

has a solution in R^n.

Suppose also that (4) implies

(5) $$cx \geq d$$

for any $x \in R^n$.

Then there is a vector $w \in R^n$ and a scalar $w_0 \in R$, with the following property:

For every $0 < \theta \leq 1$ there are nonnegative scalars $\{\lambda_i \mid i \in I\}$, only finitely non-zero, which satisfy

(6) $$c + \theta w = \sum_{i \in I} \lambda_i a^i$$

(7) $$d + \theta w_0 \leq \sum_{i \in I} \lambda_i b_i .$$

In fact, if $(v, -v_0)$ is any point in the relative interior of the set

(8) $$C'' = \text{cone}\left(\{(a^i, -b_i) \mid i \in I\} \cup \{(0,1)\}\right)$$

we may set

(9) $$(w, -w_0) = (v, -v_0) - (c, -d)$$

i.e., $w = v - c$ and $w_0 = v_0 - d$.

Proof: Since (4) is consistent, and (4) implies (5), one easily proves that

$$a^i x \geq 0, \quad \text{all } i \in I$$

implies

$$cx \geq 0.$$

Therefore for $(x,r) \in R^{n+1}$ ($x \in R^n$) arbitrary,

$$r \geq 0$$

$$a^i x - b_i r \geq 0 \quad \text{for all } i \in I$$

implies

$$cx - dr \geq 0.$$

We apply Lemma 2 to reach the conclusion, that there exists $(w, -w_0) \in R^{n+1}$, $w \in R^n$, with the following property:

For any $0 < \theta \leq 1$, there are non-negative scalars $\{\lambda_i \mid i \in I\}$, finitely non-zero, and a scalar $\varphi \geq 0$, such that

(10) $$(c,-d) + \theta(w,-w_0) = \varphi(0,1) + \sum_{i \in I} \lambda_i(a^i,-b_i).$$

Also, if $(v,-v_0)$ is any point in the relative interior of C'', we may set (9).

Now analyzing (10) by components gives (6) and (7).

Q.E.D.

To compare our results (6), (7) with the conclusions of (i), (ii), (iii) [1 , Corollary 2], divide both sides of [1 , Corollary 2(i)] by $\lambda_n > 0$ (which can be assumed by [1 , Corollary 2(ii)], to obtain that

(11) $$(c + v/\lambda_n)x \geq \theta_n/\lambda_n$$

can be "linearly deduced" (in the sense of [1]) from (4). (Note that an 'x' is missing in [1 , Corollary 2]).

Changing θ to ρ in (6), (7) of our Theorem 1, to avoid confusion with θ_n above and changing λ_i in (6), (7) to φ_i, to avoid confusion with λ_n, our result gives (with $v = w$)

(11)′ $$(c + v/\lambda_n)x = \sum_{i \in I} \varphi_i a^i x \geq \sum_{i \in I} \varphi_i b_i$$
$$\geq d + w_0/\lambda_n.$$

Thus, (11)′ can be linearly deduced from (4). Obviously, by taking $\theta_n = \lambda_n d + w_0$, from [1 , Corollary 2(ii)] we obtain [1 , Corollary 2(iii)].

This recovers [1 , Corollary 2], and in fact, more: for any sequence of $\lambda_n > 0$ with

(12) $$\lim_n \lambda_n = +\infty$$

we have seen how to construct θ_n such that [1 , Corollary 2] holds. Thus, our Theorem 1 changes an existential statement "there exists λ_n such that..." into a universal statement "for every λ_n...."

From Corollary 2, it is possible to obtain the main result of [1]. We sketch the proof. (Kortanek [7] has called this result Blair's "ascent ray" theorem.)

Theorem 2: [1]

If the system (4) is inconsistent, there exists a vector $w \in R^n$ such that, for any N, the inequality

$$wx \geq N \tag{13}$$

can be linearly deduced from (4).

Proof: The inconsistency of (4) shows that, for $(x,r) \in R^{n+1}$, $x \in R$,

$$\begin{aligned} a^i x - b_i r &\geq 0 \\ r &\geq 0 \\ \text{implies} \qquad -r &\geq 0. \end{aligned}$$

We apply Lemma 2 with $c = (0,-1)$, and find that there exists $(w,-w_0) \in R^{n+1}$, $w \in R^n$, such that (dividing (1) by θ on both sides):

For any $0 < \theta \leq 1$, there is a set of nonnegative multipliers $\{\lambda_i \mid i \in I\}$, only finitely of which are nonzero, and $\varphi \geq 0$ such that

$$(0,-1/\theta) + (w,-w_0) = \varphi(0,1) + \sum_{i \in I} \lambda_i (a^i,-b_i). \tag{14}$$

Taking components in (14), we see that

$$wx \geq 1/\theta + w_0 \tag{15}$$

can be linearly deduced from (4). If one puts $\theta = 1/(N - w_0) > 0$, the result follows, since without loss of generality, $N > w_0$ and in fact $1/(N - w_0) < 1$.

Q.E.D.

Stronger results are true for semi-infinite programs, if they are "well-behaved;" see either [5] or [6].

Section 2: A Constraint Qualification Weaker Than the Existence of a Kuhn-Tucker Vector

The convex program studied in this paper is

$$\text{(CP)} \qquad \begin{aligned} &\inf f_0(x) \\ &\text{subject to} \quad f_h(x) \leq 0 \quad \text{for all } h \in H \\ &\text{and } x \in K \end{aligned}$$

where H is an index set of arbitrary cardinality, K is a nonempty convex set, and f_h for $h \in \{0\} \cup H$ maps a convex set $D_h \supseteq K$ into R, and D_h is the domain of f_h. (In the terminology of [9], $D_h = \text{dom}(f_h)$; and $f_h(x) = +\infty$ for $x \notin D_h$ would be assumed in [9].) We also assume that $\text{relint}(K) \subseteq \text{relint}(D_h)$ for all $h \in \{0\} \cup H$, where relint(S) denotes the relative interior of the set S [9].

We introduce the following constraint qualification for (CP):

(CQ) There exists x^0 in the relative interior relint(K) of K such that $f_h(x^0) \leq 0$ for all $h \in H$.

Note that (CQ) is quite weak. E.g., for $K = R^n$, (CQ) simply asserts that (CP) is consistent. Thus, (CQ) is satisfied by the convex program

$$(16) \qquad \begin{aligned} &\inf (-y) \\ &\text{subject to} \quad (x^2 + y^2)^{1/2} - x \leq 0 \end{aligned}$$

which is well-known as not possessing any Kuhn-Tucker vector. I.e., letting v(P) denote the value of the primal problem (CP), we have here v(P) = 0 (since the constraints have solutions (x,y) = (x,0) for $x \geq 0$), yet there is

no scalar $\lambda_1 \geq 0$ with

(17) $$\inf_{x,y \in R} -y + \lambda_1[(x^2 + y^2)^{1/2} - x] \geq v(P) = 0.$$

In fact, for any $\lambda_1 > 0$, and for any specific $y_0 > 0$, by choosing $x_0 = (\lambda_1 y_0^2 - 1/\lambda_1)/2$, we have

(18) $$-y_0 + \lambda_1((x_0^2 + y_0^2)^{1/2} - x_0) = -y_0 + 1,$$

and thus the infimum in (17) is $-\infty$. For $\lambda_1 = 0$, again this infimum is $-\infty$.

The purpose of the constraint qualification (CQ), is simply to insure that the infimal value v(P) of (CP), when finite, is that of

(CP)′
$$\begin{aligned} &\inf \ \mathrm{cl}(f_0)(x) \\ \text{subject to } \ &\mathrm{cl}(f_h)(x) \leq 0 \quad \text{for all } h \in H \\ &\text{and } x \in \mathrm{cl}(K), \end{aligned}$$

where cl(S) is the closure of the set $S \subseteq R^n$, and cl(f) is the closure of the convex function f (see [9]). When this equality of value holds, since the set cl(K) and the epigraphs of the functions $\mathrm{cl}(f_h)$ for $h \in \{0\} \cup H$, can be expressed as the intersection of hyperplanes in R^{n+1} (see [9]), the techniques of the preceding section can be applied to derive results concerning v(P). From these motivational remarks, it follows that (CQ) could be replaced, in our results in the next section, by:

(CQ)′ f_h for $h \in \{0\} \cup H$ is a closed convex function, with real values in the relative interior of the convex set $D_h \supseteq K$, whose only non-real value is $+\infty$, with domain $\mathrm{dom}(f_h)$ lying between D_h and $\mathrm{cl}(D_h)$; and there exists $x^{0,h} \in \mathrm{relint}(D_h)$ with $f_h(x^{0,h}) \leq 0$.

The role of $x^{0,h}$ in (CQ)$'$ will emerge in the proof (below) of Lemma 4. To prove the claim made in our motivational remark, on the value-equivalence of (CP) and (CP)$'$ under (CQ), we establish the next result. This reproduces material from our joint paper [6], both results and proofs, with the kind permission of our co-author R.J. Duffin. Since the paper [6] is in a preliminary version, there appears to be only this course of action, given our desire to make this paper self-contained.

Lemma 3: [6] Let the value v(P) of (CP) be finite, and suppose that (CQ) holds.

Then v(P) is also the value (possibly not attained) of the convex program

$$\inf f_0(x)$$

$$\text{(19)} \qquad \text{subject to } f_h(x) \leq 0 \text{ for all } h \in H$$

$$\text{and } x \in \operatorname{relint}(K).$$

Proof: For each n, let $x^{(n)}$ be chosen to insure

$$\text{(20)} \qquad f_0(x^{(n)}) \leq v(P) + 1/n$$

$$\text{(21)} \qquad f_h(x^{(n)}) \leq 0 \quad \text{for all } h \in H$$

$$\text{(22)} \qquad x^{(n)} \in K.$$

Let $x^0 \in \operatorname{relint}(K)$ satisfy $f_h(x^0) \leq 0$ for all $h \in H$. Then, for any λ, $0 < \lambda < 1$, putting $u^{(n)} = \lambda x^{(n)} + (1-\lambda)x^{(0)}$ we have

$$\text{(23)} \qquad u^{(n)} \in \operatorname{relint}(K)$$

and by convexity

$$\text{(24)} \qquad f_h(u^{(n)}) \leq 0 \quad \text{for all } h \in H.$$

Now f_0 is continuous on the line segment $(x^{(n)}, x^0]$, with in fact

$$(25) \qquad f_0(x^{(n)}) \geq \lim \{\lambda x^{(n)} + (1-\lambda)x^{(0)} \mid 0 < \lambda < 1, \lambda \nearrow 1\},$$

as $x^0 \in \text{relint}(K) \subseteq \text{relint}(D_h)$. Hence, for $\lambda < 1$ close enough to unity,

$$(26) \qquad \begin{aligned} f_0(u^{(n)}) &\leq f_0(x^{(n)}) + 1/n \\ &\leq v(P) + 2/n \end{aligned}$$

using (20).

From (23), (24) and (26), we see that the value of the program (19) does not exceed $v(P)$. On the other hand, the program (19) is more constrained than (CP), so its value cannot be less than $v(P)$. Thus, its value is exactly $v(P)$.

Q.E.D.

Our proof of Lemma 3 also can be modified to reveal that, if $v(P) = -\infty$, then $-\infty$ is also the value of (19).

Corollary 1: If $v(P)$ is finite and (CQ) holds, then the value of (CP) is also that of (CP)′.

Proof: We recall [9] that relint(cl(K)) = relint K, and that, for all $h \in \{0\} \cup H$, $f_h(x) = (\text{cl } f_h)(x)$ for all $x \in \text{relint } K$, since relint $K \subseteq$ relint D_h is assumed.

Thus, when we start with the program (CP)′ viewed as (CP), and construct the analogous program (19), we in fact end up exactly with (19).

By Lemma 3, both (CP) and (CP)′ have the value of (19), hence both have value $v(P)$.

Q.E.D.

Section 3: The Main Result

Our main result (Theorem 3) is obtained by applying Theorem 1 to a "semi-infinite" system of linear inequalities equivalent to $(CP)'$, and then interpreting this outcome by methods of algebraic manipulation developed in [6]. We reproduce the latter here, for the sake of a self-contained presentation.

By Corollary 1, we may assume, within our proofs, that K is closed, as are the functions f_h for $h \in \{0\} \cup H$. This is assumed throughout the remainder. Therefore, we have representations via hyperplanes:

$$K = \{x \in R^n \mid a^j x \geq a_0^j,\ j \in I(-1)\} \tag{27}$$

$$\mathrm{epi}(f_h) = \{(z,x) \in R^{n+1},\ x \in R^n \mid b_0^j z + a^j x \geq a_0^j,\ j \in I(h)\} \tag{28$_h$}$$

for index sets $I(h)$, $h \in \{-1\} \cup \{0\} \cup H$, where possibly $I(-1) = \emptyset$, but $I_h \neq \emptyset$ for $h \neq -1$, and as usual $\mathrm{epi}(f_h)$ denotes the epigraph of f_h:

$$\mathrm{epi}(f_h) = \{(z,x) \in R^{n+1},\ x \in R^n \mid z \geq f_h(x)\} \tag{29}$$

which is a closed, convex set. Obviously, in $(28)_h$, $b_0^j \geq 0$ for all $j \in I(h)$ and $h \in \{0\} \cup H$.

Without loss of generality, the representations (27), $(28)_h$ consist precisely of all supporting hyperplanes for the closed, convex sets K and $\mathrm{epi}(f_h)$, $h \in \{0\} \cup H$.

Lemma 4: [6] Fix $h \in H$, and suppose that (CQ) holds.

Then for any $x \in D_h$, $f_h(x) \leq 0$ is equivalent to the semi-infinite system

$$a^j x \geq a_0^j,\quad j \in I(h). \tag{30}$$

Proof: If $f_h(x) \leq 0$, since $(f_h(x),x) \in \mathrm{epi}(f_h)$ and $b_0^h \geq 0$ for all $j \in I(h)$, we have

$$a^j x \geq a_0^j + (-b_0^j)f_h(x) \geq a_0^j$$

whenever $x \in D_h$ and $j \in I(h)$.

For the converse, suppose (30) holds for $\bar{x}$ (i.e., $a^j\bar{x} \geq a_0^j$ for all $j \in I(h)$) yet $f_h(\bar{x}) > 0$. By (CQ), there exists $x^0 \in \mathrm{relint}(K) \subseteq \mathrm{relint}(D_h)$ with $f_h(x^0) \leq 0$. Without loss of generality, f_h is closed. Therefore, it is continuous on line segments and, denoting $x(\lambda) = \lambda\bar{x} + (1-\lambda)x^0$, $0 \leq \lambda \leq 1$, we have $f_h(\bar{x}) = \lim \{x(\lambda) \mid \lambda \nearrow 1\}$. Thus, for some λ^*, with $0 < \lambda^* < 1$, $f_h(x(\lambda^*)) > 0$.

For all λ in the range $0 \leq \lambda < 1$, $x(\lambda) \in \mathrm{relint}(K) \subseteq \mathrm{relint}(D_h)$, and hence there exists a subgradient $-u \in R^n$ to f_h at $x(\lambda^*)$, i.e.,

$$f_h(x) \geq f_h(x(\lambda^*)) + u(x(\lambda^*) - x) \quad \text{for all } x \in D_h. \tag{32a}$$

From $f_h(x(\lambda^*)) > 0$ and $f_h(x^0) = f_h(x(0)) \leq 0$, we obtain from (32a) (with $x = x^0$), $0 \geq f_h(x(\lambda^*)) + \lambda^* u(\bar{x} - x^0)$ and hence $u(\bar{x} - x^0) < 0$. This in turn implies that $u(\bar{x} - x(\lambda^*)) = u(\bar{x} - \lambda^*\bar{x} - (1-\lambda^*)x^0) = u(1-\lambda^*)(\bar{x} - x^0) < 0$. Thus, since $f_h(x(\lambda^*)) > 0$, the inequality $u\bar{x} \geq f_h(x(\lambda^*)) + ux(\lambda^*)$ cannot hold.

Now, since (32a) holds, without loss of generality,

$$z + ux \geq f_h(x(\lambda^*)) + ux(\lambda^*) \tag{32b}$$

is among the defining inequalities in the system $(28)_h$. I.e., for some $j \in I(h)$, $b_0^j = 1$, $a^j = u$, and $a_0^j = f_h(x(\lambda^*)) + ux(\lambda^*)$. However, since $u\bar{x} < f_h(x(\lambda^*)) + ux(\lambda^*)$, as we saw in the last paragraph, for this choice of $j \in I(h)$ we have $a^j\bar{x} < a_0^j$, and this contradicts (30) and completes the proof.

Q.E.D.

Corollary 2: (Á là [6]).

Assume that (CQ) holds. Then the value v(P) of the convex program, if finite, is also the value of this semi-infinite program:

$$\text{(SI)} \qquad \begin{aligned} &\inf z \\ &\text{subject to } b_0^j z + a^j x \geq a_0^j, \quad j \in I(0) \\ &\qquad a^j x \geq a_0^j \quad \text{for all } j \in I(h) \\ &\qquad\qquad \text{and } h \in \{-1\} \cup H. \end{aligned}$$

Proof: Immediate from Lemma 4. Q.E.D.

It is now the point to complete the program outlined at the beginning of this paper, and invoke Theorem 1. This program is identical in conception to that of our joint paper [6], which differed only in that a different and stronger result on the semi-infinite system (SI) was invoked, which was possible in [6] because a constraint qualification stronger than (CQ) was assumed, which led to an (SI) with some special properties.

Theorem 3: Suppose that the constraint qualification (CQ) holds for the convex program (CP) of finite value v(P).

Then there exists w_0, $w_1 \in R$ and $w \in R^n$ with the following property:

For any scalar θ in the range $0 < \theta \leq 1$, there exist $\gamma \in R^n$, $\gamma^0 \in R$, and nonnegative scalars $\{\lambda_h \mid h \in H\}$, only finitely many of which are nonzero, and $\beta^h \in R^n$, $\beta_0^h \in R$ for $h \in \{0\} \cup H$, satisfying:

<u>Condition 1</u>: $\gamma x + \gamma_0 \leq 0$ for $x \in K$

<u>Condition 2</u>: $\beta^h x + \beta_0^h \leq f_h(x)$ for all $x \in D_h$ and $h \in \{0\} \cup H$

<u>Condition 3</u>:

$$\gamma x + \gamma^0 + (1+\theta w_0)(\beta^0 x + \beta_0^0) + \theta(wx + w_1) + \sum_{h \in H} \lambda_h(\beta^h x + \beta_0^h) \geq v(P)$$

for all $x \in R^n$.

In fact, w_0, w_1, w can be chosen arbitrarily to satisfy

$$(33) \qquad (w_0, w, w_1) = (v_0, v, v_1) - (1, 0, -v(P))$$

where (v_0, v, v_1) (with $v_1, v_2 \in R$ and $v \in R^n$) is any point in the relative interior of

$$(34) \qquad C = \text{cone}\left(\{(b_0^j, a^j, -a_0^j) \mid j \in I(0)\} \cup \bigcup_{h \in \{-1\} \cup H} \{(0, a^j, -a_0^j) \mid j \in I(h)\} \cup \{(0,0,1)\}\right).$$

<u>Proof</u>: By Corollary 2 and by Theorem 1, equations (6) and (7) (which we saw is equivalent to (10)), we have, for $0 < \theta \leq 1$,

$$(35) \qquad (1, 0, -v(P)) + \theta(w_0, w, w_1) = \varphi(0,0,1) + \sum_{j \in I(0)} \varphi_j(b_0^j, a^j, -a_0^j) + \sum_{h \in \{-1\} \cup H} \sum_{j \in I(h)} \varphi_j(0, a^j, -a_0^j),$$

since (SI) implies $z \cdot 1 + x \cdot 0 \geq v(P)$. Of course, in (35), $\varphi \geq 0$ and all $\varphi_j \geq 0$, and only finitely many of the quantities φ_j for all $j \in I(h)$ and all $h \in \{-1, 0\} \cup H$ are actually nonzero. Also, (w_0, w, w_1) is any solution

to (33), such that (v_0, v, v_1) is in the relative interior of C of (34).

We now analyze (35) by the methods of [6].

From the first components in (35), we obtain

$$1 + \theta w_0 = \sum_{\substack{j \in I(0) \\ b_0^j > 0}} \varphi_j b_0^j. \tag{36}$$

We will, in general, define for $h \in \{0\} \cup H$

$$\lambda_h = \sum_{\substack{j \in I(h) \\ b_0^j > 0}} \varphi_j b_0^j \tag{37}$$

(with the understanding that $\lambda_h = 0$ if $b_0^j = 0$ for all $j \in I(h)$ with $\varphi_j > 0$) for $h \in \{0\} \cup H$. Thus we have $\lambda_0 = 1 + \theta w_0$ from (36). Clearly, only finitely many λ_h can be nonzero, and all $\lambda_h \geq 0$, by the conditions on the scalars $\varphi_j \geq 0$.

Next we define these vectors and scalars:

$$\gamma = -\sum_{j \in I(-1)} \varphi_j a^j - \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b_0^j = 0}} \varphi_j a^j \tag{38a}$$

$$\gamma_0 = \sum_{j \in I(-1)} \varphi_j a_0^j + \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b_0^j = 0}} \varphi_j a_0^j \tag{38b}$$

where an empty summation is zero. From (27), if $j \in I(-1)$, $a^j x \geq a_0^j$ for $x \in K$; and from $(28)_h$, if $b_0^j = 0$, we have again $a^j x \geq a_0^j$ for $j \in I(h)$ and $h \in \{0\} \cup H$, as $x \in D_h \supseteq K$. Hence $\gamma x + \gamma_0 \leq 0$ for $x \in K$, i.e., Condition 1 holds.

For $h \in \{0\} \cup H$, if $b_0^j > 0$, since $b_0^j f_h(x) + a^j x \geq a_0^j$ for $x \in D_h$, we have

$$f_h(x) \geq -\frac{a^j}{b_0^j} x + a_0^j / b_0^j, \quad \text{for } x \in D_h. \tag{39}$$

Combining (37) and (39) for $\lambda_h > 0$, we have

$$\begin{aligned} \lambda_h f_h(x) &= \sum_{\substack{j \in I(h) \\ b_0^j > 0}} \varphi_j b_0^j f_h(x) \\ &\geq \sum_{\substack{j \in I(h) \\ b_0^j > 0}} \varphi_j b_0^j \left(-\frac{a^j}{b_0^j} x + a_0^j / b_0^j\right), \quad \text{for } x \in D_h, \end{aligned} \tag{40}$$

and so defining

$$\beta^h x + \beta_0^h = \frac{1}{\lambda_h} \sum_{\substack{j \in I(h) \\ b_0^j > 0}} \varphi_j b_0^j \left(-\frac{a^j}{b_0^j} x + a_0^j / b_0^j\right) \tag{41}$$

we obtain Condition 2. To be precise, we actually have $\mathrm{cl}(f_h)(x) \geq \beta^h x + \beta_0^h$ for all $x \in D_h$, but since $\mathrm{cl}(f_h)(x) \leq f_h(x)$ for all $x \in D_h$, Condition 2 follows.

If $\lambda_h = 0$, one can arbitrarily pick $\beta^h x + \beta_0^h$ to satisfy Condition 2, at least one such affine form existing since f_h is somewhere finite, and hence has at least one subgradient at one point.

For the concluding part of our analysis, we write (35) again with the first component dropped, in this form:

$$\begin{aligned} (0, -v(P)) + \theta(w, w_1) &= \varphi(0,1) \\ &+ (-\gamma, -\gamma_0) \\ &+ \sum_{h \in \{0\} \cup H} \sum_{\substack{j \in I(h) \\ b_0^j > 0}} \varphi_j b_0^j (a^j / b_0^j, -a_0^j / b_0^j) \end{aligned} \tag{35$'$}$$

$$= \varphi(0,1) + (-\gamma,-\gamma_0)$$

$$+ \sum_{h\in\{0\}\cup H} \lambda_h(-\beta^h,-\beta_0^h).$$

It remains only to dot product both sides of (35)′ with (-x,-1), where $x \in R^n$ is arbitrary, to obtain

(42) $$v(P) + \theta(-wx - w_1) = -\varphi + (\gamma x + \gamma_0)$$

$$+ \sum_{h\in\{0\}\cup H} \lambda_h(\beta^h x + \beta_0^h)$$

for all $x \in R^n$. Since $\varphi \geq 0$, (42) immediately yields Condition 3, using $\lambda_0 = 1 + \theta w_0$.

Q.E.D.

<u>Corollary 3</u>: Suppose that the constraint qualification (CQ) holds for the convex program (CP) of finite value v(P).

Then there exists w_0, $w_1 \in R$ and $w \in R^n$ with the following property:

For any scalar in the range $0 < \theta \leq 1$, there exist scalars $\{\lambda_h \mid h \in H\}$, only finitely many of which are nonzero, with

(43) $$(1 + \theta w_0)f_0(x) + \theta(wx + w_1) + \sum_{h\in H} \lambda_h f_h(x) \geq v(P)$$

for all $x \in K$.

Furthermore, w_0, w_1 and w can be arbitrarily chosen, subject to the condition (33), where (v_0,v,v_1) is a point in the relative interior of the convex set C of (34).

<u>Proof</u>: Immediate from Theorem 3.

Q.E.D.

Section 4: Relationship to the Usual Lagrangean

We conclude by attending to two matters, which are not of great technical difficulty, but which do serve to clarify the nature of our main result (Theorem 3), more specifically its corollary (Corollary 3), and how, in a limiting sense, Corollary 3 solves (CP). This clarification involves some specification on the vectors (w_0, w, w_1) of Corollary 3 in certain cases.

In the usual Lagrangean and its associated Kuhn-Tucker theory, typically one seeks sufficient conditions for the equality,

$$(44) \qquad \max_{\substack{\lambda_h \geq 0 \\ h \in H}} \ \inf_{x \in K} \ \{f_0(x) + \sum_{h \in H} \lambda_h f_h(x)\} = v(P),$$

where, in some instances, the "max" is relaxed to a "sup" (supremum). The usual theory as developed in [9] requires the cardinality of H to be finite. As we saw in Section 2, our constraint qualification (CQ) does not even insure (44) with "max" replaced by "sup."

Our additional conditions $D_h \supseteq K$ and $\mathrm{relint}(D_h) \supseteq \mathrm{relint}(K)$ are also typical of the standard treatment of "ordinary convex programs," as presented in [9], and in fact [9] requires $D_0 = K$, which we do not. (Restricting a function f_0 from a domain $D_0 \supsetneq K$ to K will typically significantly affect its subgradients. E.g., a one-dimensional differentiable convex function defined on R has a unique subgradient at x = 0, but if f is restricted to $K = \{x \mid x \geq 0\}$ it has many subgradients at zero. It is essentially due to this restriction $D_0 = K$, that no term of the form $\gamma x + \gamma_0$, nonpositive on K, appears in certain results of [9], such as [9 , Theorem 28.3 and particularly Theorem 28.3(c)].

What our Corollary 3 concerns, is relations more complex than (44), due to the presence of one more operation on the left-hand-side: the taking of a limit.

In what follows Λ denotes all multipliers $(\lambda_h \mid h\in H)$ which are nonnegative and finitely nonzero. We next establish a crucial inequality.

Lemma 5: Suppose that $(w_0(\theta), w(\theta), w_1(\theta)) \in R^{n+2}$, $w(\theta) \in R^n$, is defined for $0 < \theta \leq 1$, and the set of all vectors of this form is bounded. Then if (CP) is consistent and has finite value v(P),

$$(45) \qquad \limsup_{\theta \searrow 0^+} \sup_{\Lambda} \inf_{x\in K} \{(1+\theta w_0(\theta))f_0(x) + \theta(w(\theta)x + w_1(\theta)) + \sum_{h\in H} \lambda_h f_h(x)\} \leq v(P)$$

Proof: First observe that for any θ, and element of Λ,

$$(46) \qquad \begin{aligned} &\inf_{x\in K} \{(1+\theta w_0(\theta))f_0(x) + \theta(w(\theta)x + w_1(\theta)) + \sum_{h\in H} \lambda_h f_h(x)\} \\ &\leq \inf_{x\in K} \{(1+\theta w_0(\theta))f_0(x) + \theta(w(\theta)x + w_1(\theta)) \mid f_h(x) \leq 0,\ h\in H\} \end{aligned}$$

since all $\lambda_h \geq 0$ if $(\lambda_h \mid h\in H) \in \Lambda$. Therefore, the left-hand-side in (45) does not exceed

$$(47) \qquad \limsup_{\theta \searrow 0^+} \inf_{x\in K} \{(1+\theta w_0(\theta))f_0(x) + \theta(w(\theta)x + w_1(\theta)) \mid f_h(x) \leq 0,\ h\in H\}.$$

Next, let $x^{(n)}$ be chosen so that $f_h(x^{(n)}) \leq 0$ for $h \in H$ and $f_0(x^{(n)}) \leq v(P) + 1/n$, which is possible since v(P) is the value of (CP). We see that (47) does not exceed

$$(48) \qquad \limsup_{\theta \searrow 0^+} \{(1+\theta w_0(\theta))f_0(x^{(n)}) + \theta(w(\theta)x^{(n)} + w_1(\theta))\}$$

$$= f_0(x^{(n)}) < v(P) + 1/n,$$

using the boundedness condition on $(w_0(\theta), w(\theta), w_1(\theta))$. Now if $v(P) + 1/n$ is an upper bound on the left-hand-side of (45) for any n, so is v(P). This establishes (45).

Q.E.D.

From (43) of Corollary 3, for any θ in the range $0 < \theta \leq 1$ we have

$$(49) \qquad \sup_{\Lambda}{}^{+} \inf_{x \in K} \{(1+\theta w_0)f_0(x) + \theta(wx + w_1) + \sum_{h \in H} \lambda_h f_h(x)\} \geq v(P),$$

where a statement $\sup^{+} \{\alpha_j \mid j \in J\} \geq \alpha$ for us abbreviates the condition that $\alpha_j \geq \alpha$ for some $j \in J$, and hence

$$(49)' \qquad \liminf_{\theta \searrow 0^+} \sup_{\Lambda}{}^{+} \inf_{x \in K} \{(1+\theta w_0)f_0(x) + \theta(wx + w_1) + \sum_{h \in H} \lambda_h f_h(x)\} \geq v(P).$$

Putting together (45) of Lemma 5 and (49)′ above, we obtain (under the hypotheses of Corollary 3)

$$(50) \qquad \lim_{\theta \searrow 0^+} \sup_{\Lambda}{}^{+} \inf_{x \in K} \{(1+\theta w_0)f_0(x) + \theta(wx + w_1) + \sum_{h \in H} \lambda_h f_h(x)\} = v(P).$$

Comparing the standard Lagrangean result (44) with ours (50), we see their similarity in nature. The limit appearing in (50) suggests the term "limiting Lagrangean" for our results.

We now state a way in which our limiting Lagrangean can be used to solve (CP), in a limiting sense.

Lemma 6: Suppose that $(w_0(\theta), w(\theta), w_1(\theta)) \in R^{n+2}$, $w(\theta) \in R^n$, is defined for $0 < \theta \leq 1$, and the set of all vectors of this form is bounded. Let the value v(P) of (CP) be finite.

Let θ_n in the range $0 < \theta_n \leq 1$ satisfy

(51) $$\lim_n \theta_n = 0 .$$

Suppose also that the functions f_h, $h \in \{0\} \cup H$ are continuous, and that for each n the vector $x^{(n)}$ satisfies, for certain $(\lambda_h \mid h \in H) \in \Lambda$,

(52) $$v(P) \leq \inf_{x \in K} \{(1+\theta_n w_0(\theta_n))f_0(x) + \theta_n(w(\theta_n)x + w_1) + \sum_{h \in H} \lambda_h^n f_h(x)\}$$

$$\leq (1+\theta_n w_0(\theta_n))f_0(x^{(n)}) + \theta_n(w(\theta_n)x^{(n)} + w_1) + \sum_{h \in H} \lambda_h^n f_h(x^{(n)})\} + \sigma_n$$

with $\limsup_n \sigma_n \leq 0$

and

(53) $$f_h(x^{(n)}) \leq \sigma_n^h, \text{ with } \limsup_n \sigma_n^h \leq 0 \text{ for all } h \in H;$$

(54) $$\sum_{h \in H} \lambda_h^n f_h(x^{(n)}) = \rho_n \text{ where } \limsup_n \rho_n = 0.$$

Then if the sequence of $x^{(n)}$ has a limit point x^*, x^* will be an optimum to (CP).

<u>Proof</u>: By continuity, as we may assume $\lim_n x^{(n)} = x^*$, we have $f(x^*) = \lim_n f_h(x^{(n)}) \leq \limsup_n \sigma_n^h \leq 0$, so x^* is feasible in (CP).

By virtually repeating the proof of Lemma 5, we obtain, as $\limsup_n \sigma_n \leq 0$,

(55) $$\limsup_n \sup_\Lambda \{(1+\theta_n w_0(\theta_n))f_0(x^{(n)}) + \theta_n(w(\theta_n)x^{(n)} + w\ (\theta_1))$$

$$+ \sum_{h \in H} \lambda_h f_h(x^{(n)})\} \leq v(P).$$

Using (51), (54), and (55) we obtain (via the boundedness assumption)

$$(56)\qquad v(P) \geq \limsup_{n} \{(1+\theta_n w_0(\theta_n))f_0(x^{(n)}) + \theta_n(w(\theta_n)x^{(n)} + w_1(\theta_n))\}$$

$$= f(x^*).$$

Thus, $f(x^*)$ is optimal in (CP).

Q.E.D.

As regards the determination of $w_0 \in R$ and $w \in R^n$ of Theorem 3 and Corollary 3, we first discuss a few points concerning the determination of interior points (v_0,v,v_1) of C in (34). We do not try here to give an efficient algorithm; we merely wish to indicate that these quantities are often,in principle, computable. Since $w_1 = v_1 + v(P)$, the determination of w_1 involves a knowledge of the value v(P) of the program (CP), and hence is a more complex matter. We shall make remarks about w_1 of (33) in our concluding statements.

For vectors $v^i \in R^q$ and a nonempty index set $I \neq \emptyset$, the determination of an interior point of

$$(57)\qquad C_I = \text{cone}\,(\{v^i \mid i \in I\})$$

is never, in principle, problematic, once one has some spanning set, say $\{v^1,\ldots,v^t\}$ of $\{v^i \mid i \in I\}$, in the sense of a vector space span. An interior point of C_I of (57) is always given by

$$(58)\qquad v = v^1 + \ldots + v^t.$$

Indeed, for any vector $w = \sum_{i=1}^{t} \rho_i v^i$ in the vector space spanned by $\{v^1,\ldots,v^t\}$, i.e., in the manifold spanned by C_I of (57), there is a

sufficiently small $\varepsilon > 0$ so that

$$(59) \qquad 1 + \varepsilon\rho_i > 0 \quad \text{for } i = 1,\ldots,t$$

and hence $v + \varepsilon w \in C_I$. Since w in the manifold spanned by C_I was arbitrary, v of (58) is an interior point of C_I, by standard criteria (see e.g., [9]).

To obtain $\{v^1,\ldots,v^t\}$, it often suffices to know t, the dimension of the manifold spanned by C_I. Here we have in mind primarily the case that a countable dense subset $\{v^i \mid i\in I\}$, which can be effectively listed, can be effectively extracted from $\{v^i \mid i\in I\}$. Since clcone $(\{v^i \mid i\in I'\})$ $\supseteq$ cone $(\{v^i \mid i\in I\})$, there are also t linearly independent vectors in $\{v^i \mid i\in I'\}$. Thus one can simply continue a listing until t independent ones are found. We avoid details on the points raised in this paragraph, since a full discussion of these matters requires a knowledge of recursion theory, which we do not assume here.

The simplest case is $t = q$, i.e., the cone C_I of (57) is fully-dimensional, and our next result shows that this is indeed a very common case for the cone C of (34). Before we begin the proof of our next result, one may remark that full-dimensionality occurs if

$$(60) \qquad v^i x = 0 \quad \text{for all } i \in I \quad \text{implies } x = 0.$$

Indeed, if (60) held but the linear span of $\{v^i \mid i\in I\}$ was a subspace $L \subsetneq R^q$, *its perpendicular subspace* $L^\perp$ has a nonzero vector $\bar{x}$. Then $v^i\bar{x} = 0$ for all $i \in I$ but $\bar{x} \neq 0$, a contradiction.

<u>Lemma 7</u>: Suppose that (CP) is feasible and has finite value $v(P)$. Barring the case, that there exists a nonzero vector x^* such that for all $\theta \in R$, we have for any solution $\bar{x}$ to (CP):

(62a) $$f_h(\bar{x} + \theta x^*) \leq 0 \quad \text{all } h \in H;$$

(62b) $$f_0(\bar{x} + \theta x^*) = f(\bar{x});$$

(62c) $$\bar{x} + \theta x^* \in K;$$

then the cone C of (34) is of full dimension $(n+2)$.

<u>Proof</u>: We have to show that there is no nonzero solution to all the equalities

(63) $$\begin{aligned} b_0^j z + a^j x - a_0^j w &= 0, \quad j \in I(0) \\ a^j x - a_0^j w &= 0, \quad j \in I(h) \text{ and } h \in \{-1\} \cup H \\ w &= 0 \end{aligned}$$

or, equivalently, to the equalities

(63)′ $$\begin{aligned} b_0^j z + a^j x &= 0, \quad j \in I(0) \\ a^j x &= 0, \quad j \in I(h) \text{ and } h \in \{-1\} \cup H. \end{aligned}$$

Since at least one $b_0^j > 0$ for $j \in I(0)$, as f_0 has a subgradient at at least one point, we cannot have $x = 0$ in a nonzero solution to (63)′.

Suppose that (z^*,x^*) solves (63)′, so that $x^* \neq 0$. By homogeneity, we may assume $z^* \leq 0$. Let $\bar{x}$ be any solution to (CP). Then for any $\theta \geq 0$, fixing $h \in \{-1\} \cup H$ and letting $j \in I(h)$ be arbitrary, we have

(69) $$\begin{aligned} a^j(\bar{x} + \theta x^*) &= a^j\bar{x} + \theta a^j x^* \\ &= a^j\bar{x} \geq a_0^j \end{aligned}$$

since $a^j x^* = 0$ from (63)′ and $a^j\bar{x} \geq a_0^j$ by Lemma 4. Thus, by Lemma 4,

we have (62a) and (62c) for $\theta \geq 0$. Also, since $b_0^j f(\bar{x}) + a^j\bar{x} \geq a_0^j$ for any $j \in I(0)$ (as $(f_0(x), x) \in \mathrm{epi}(f_0)$), we have

$$b^j(f_0(\bar{x}) + \theta z^*) + a^j(\bar{x} + \theta x^*) \tag{65}$$

$$= (b^j f_0(\bar{x}) + a^j\bar{x}) + \theta(b^j z^* + a^j x^*)$$

$$= b^j f_0(\bar{x}) + a^j\bar{x} \geq a_0^j$$

since $b^j z^* + a^j x^* = 0$. This gives (62b) for $\theta \geq 0$, with = replaced by $\leq$, since $(f_0(\bar{x}) + \theta z^*, \bar{x} + \theta x^*) \in \mathrm{epi}(f_0)$ and $z^* \leq 0$, $f_0(\bar{x}) + \theta z^* \leq f(\bar{x})$ for all $\theta \geq 0$.

Now if $z^* < 0$, from the above, $f_0(\bar{x} + \theta x^*)$ can be indefinitely decreased by sending $\theta \nearrow +\infty$, and all the while $\bar{x} + \theta x^*$ is feasible in (CP). This contradicts that CP has finite value. Hence $z^* = 0$, and we can repeat the analysis with $(-z^*, -x^*)$ replacing (z^*, x^*), and in this manner obtain (62a) and (62c) for all $\theta \in R$. We also obtain $f_0(\bar{x} + \theta x^*) \leq f_0(\bar{x})$ for all $\theta \in R$; since f_0 is convex, we clearly have (62b).

Q.E.D.

Thus, if it is known that the feasible region contains no full line, or that $|f(x)| \nearrow +\infty$ as $\|x\| \nearrow +\infty$, or that f_0 is not constant on any line, or that f_0 is not constant on any line in the feasible region of (CP) — all of these being commonly-occurring hypotheses — Lemma 7 shows that the dimension of the cone C of (42) is full, i.e., is $(n+2)$.

From (33), once an interior point (v_0, v, v_1) is found, we can compute (w_0, w, w_1) for use in the limiting Lagrangean by $w_0 = v_0 - 1$, $w = v$, and $w_1 = v_1 + v(P)$. Only the last equation is problematic, since it appears to involve an exact knowledge of the value $v(P)$ of the convex program (CP).

However, an inspection of (43) shows that, if $w_1' \geq w_1$ replaces w_1 in the limiting Lagrangean, we will still obtain a limiting Lagrangean, as $\theta \to 0$. Therefore it is necessary only to know a bound M on the value of (CP), and since $v(P) \leq M$, we may set $w_1' = v_1 + M$.

Often such bounds M are obtained from feasible solutions to (CP). In any event, since M can be set most liberally, even the most cursory information about (CP) will allow one to compute w_1' from v_1.

A few final remarks are in order. The limiting Lagrangean can, of course, reduce to the ordinary one, if $(1,0,-v(P))$ is in the relative interior of the cone C of (34), for then we get $(w_0,w,w_1) = 0$ in (33). However, it is possible that

$$(66) \qquad (1,0,-v(P)) \in C$$

but not $(1,0,-v(P)) \in \text{relint } C$ (recall that always $(1,0,-v(P)) \in cl(C)$), in which case the limiting Lagrangean does not reduce to the ordinary one.

Nevertheless, in this latter case, one easily sees that the term $\theta(w_0,w,w_1)$ can be omitted in (35), and then if the analysis in the remainder of the proof of Theorem 3 is repeated, we obtain the usual Lagrangean-type relationship (44). We say "Lagrangean-type" rather than "Lagrangean," since H can be infinite, yet only finitely many of the λ_h, $h \in H$ will actually be nonzero (i.e., all but finitely-many of the constraints of (CP) can be omitted without changing its value).

To get Lagrangean results of the usual type (44), one needs to know when (66) holds. In this regard, we have jointly verified [6] that many of the "constraint qualifications" (such as those of [9 , Theorem 28.2]

and others) are sufficient conditions for the cone C of (34) to be closed, provided that the "correct" defining inequalities are chosen in equations (27), $(28)_h$. (An "arbitrary" representation of K or $\mathrm{epi}(f_h)$, which simply defines the correct set of points, usually will not do: the representation plays a role as important as the actual set. For a similar circumstance where this issue arose, see [3] and [5].)

The closure of C trivially implies (66), by Lemma 1, as $(1,0,-v(P)) \in \mathrm{cl}(C)$. And since a "constraint qualification" is a condition on the constraints alone, allowing the objective function f_0 to vary over all convex functions, it is not surprising that the closure of C is implied by these hypotheses. More can be said on these matters; see [6] for further details and results.

April 26, 1978

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We show, for convex optimization in R^n, how a minor modification of the usual Lagrangean function (unlike that of the "augmented Lagrangeans"), plus a limiting operation, allows one to close duality gaps even in the absence of a Kuhn-Tucker vector (see the introductory discussion, and the discussion in Section 4 up to equation (50)). The cardinality of the convex constraining functions can be arbitrary (finite, countable, or uncountable).

(over)

In fact, our main result (Theorem 3 of Section 3) reveals much finer detail concerning our "Limiting Lagrangean." There are affine minorants (for any value $0 < \theta \leq 1$ of the limiting parameter θ) of the given convex functions, plus an affine form nonpositive on K, for which a general linear inequality holds on R^n. After substantial weakening, this inequality leads to the conclusions of the previous paragraph.

This work is motivated by, and is a direct outgrowth of, research joint with R.J. Duffin, which is cited as our reference [6].